SCATTERING OF PLANE SH-WAVES BY A CIRCULAR CAVITY IN A PRE-STRESSED ELASTIC MEDIUM

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Abstract: Fundamental studies on elastic waves propagating in the presence of obstacles such as cavities, embedded footings and underground tunnels are important in earthquake engineering. Previous studies have been restricted to the case where the surrounding medium is considered to be linear elastic. In this paper, the effect of pre-stress on the scattering of plane SH-waves from a circular cylindrical cavity in a compressible isotropic elastic medium, is studied. The complex function method is employed to analyze the incremental boundary value problem. The spatial variables (x_1, x_2) are mapped on to two different complex planes, to represent the series solution of the incident waves and the scattered waves. The coefficient of each term in the series solution can be computed numerically from a set of linear simultaneous equations, which are constructed by satisfying the incremental traction free boundary condition along the surface of the cavity. Varga material is assumed in the numerical example and in the absence of pre-stress, the analytical solution for linear isotropic elastic case is recovered. Varying the values of principal stretches, the effect of pre-stress on the speed of incident SH-waves and the dynamic stress concentration factor along the surface of the cavity is clearly seen.

1. INTRODUCTION

The dynamic response due to seismic waves, of underground structures such as foundations of superstructures, transportation tunnels or pipelines, is an important engineering problem. There are many previous studies on the analysis of wave scattering problems, where linear elastic assumptions are used, e.g., analysis of SH-waves scattered from a rigid semicircular cylinder embedded in isotropic half-space (Wijeyewickrema and Keer, 1986) and dynamic stress concentration analysis for SH-wave scattering from a cavity in a linear anisotropic elastic medium (Liu 1988; Liu and Han, 1993; Han et. al. 1995). However, considering that the earth is an initially stressed medium, it may be more appropriate to model the earth as a pre-stressed elastic medium. In the last two decades, wave propagation problems in pre-stressed elastic media have been extensively studied but due to the complexity of the analysis, which comes from the effects of pre-stress, analytical results of wave reflection and scattering problems, have been limited to reflection of waves from a linear plane boundary or interface only (Ogden and Sotiropoulos, 1998 and Hussain and Ogden, 2001).

In the present paper, SH-wave scattering from a circular cylindrical cavity in a compressible pre-stressed unbounded elastic medium is studied. The complex function method is employed where the fundamental equations and formulation of the problem are given in Sec. 2. The numerical results of two examples when the Varga strain energy function is assumed are given in Sec. 3, and the dynamic stress concentration factor and the non-dimensional displacement and stresses are plotted. The conclusions in Sec. 4 are useful for engineering practice and further studies are suggested.

2. FORMULATION OF THE PROBLEM

2.1 Basic Equations

Consider a homogeneous compressible isotropic elastic material with an initial unstressed state denoted by B_u, which after being subjected to pure homogeneous strains has the new configuration B_e, the pre-stressed equilibrium state. A Cartesian co-ordinate system $Ox_1x_2x_3$, with axes coincident with the principal axes of strain, is chosen for the configuration B_e. Let $\mathbf{u}$ be a small, time dependent displacement superimposed on B_e. The incremental equations of motion for small time dependent displacements superimposed on the finite quasi-static deformation and the component of incremental nominal stress tensor $\mathbf{s_0}$, can be written as (Chapter 6, Ogden, 1984)

$$\mathrm{A}_{0jikl}u_{l,jk} = \rho \ddot{u}_i, \qquad s_{0ji} = \mathrm{A}_{0jilk}u_{k,l}, \tag{1}$$

where A_{0jilk} are the components of the fourth-order tensor of first-order instantaneous elastic moduli which relates the nominal stress increment tensor and the deformation gradient increment tensor, ρ is the material density in the current configuration and superimposed dot and comma indicate differentiation with respect to time t and spatial coordinate component in B_e, respectively.

The corresponding equations for anti-plane deformation where $u_3 = u_3(x_1, x_2, t)$ and $u_1 = u_2 = 0$ can be written as

$$\mathrm{A}_{01313}u_{3,11} + \mathrm{A}_{02323}u_{3,22} = \rho \ddot{u}_3, \quad s_{013} = \mathrm{A}_{01313}u_{3,1}, \quad s_{023} = \mathrm{A}_{02323}u_{3,2} \tag{2}$$

in which the instantaneous elastic moduli A_{01313} and A_{02323} are given in term of the strain energy function $W(\lambda_1, \lambda_2, \lambda_3)$ and the principal stretches $\lambda_i, (i = 1, 2, 3)$ as,

$$J\mathrm{A}_{0i3i3} = \begin{cases} (\lambda_i W_i - \lambda_3 W_3)\lambda_i^2/(\lambda_i^2 - \lambda_3^2), & \lambda_i \neq \lambda_3 \\ \frac{1}{2}(\lambda_i^2 W_{ii} - \lambda_i\lambda_3 W_{i3} + \lambda_i W_i), & \lambda_i = \lambda_3, \end{cases} \tag{3}$$

where $W_i = \partial W/\partial \lambda_i$, $W_{ij} = \partial^2 W/\partial\lambda_i\partial\lambda_j$ and $J = \lambda_1\lambda_2\lambda_3$ (Roxburgh and Ogden, 1994).

The incremental nominal stress components s_{0r3} and $s_{0\theta 3}$ in the cylindrical coordinate system (r, θ, x_3) where $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ can be expressed as,

$$s_{0r3} = \mathrm{A}_{01313}u_{3,1}\cos\theta + \mathrm{A}_{02323}u_{3,2}\sin\theta, \quad s_{0\theta 3} = -\mathrm{A}_{01313}u_{3,1}\sin\theta + \mathrm{A}_{02323}u_{3,2}\cos\theta. \tag{4}$$

Since the complex function method is used from Eq. (4), the complex expression of non-dimensional stresses are obtained as,

$$\hat{s}_{0r3} - i\hat{s}_{0\theta 3} = [\gamma^2 u_{3,1} - iu_{3,2}]\mathrm{e}^{i\theta}, \quad \hat{s}_{0r3} + i\hat{s}_{0\theta 3} = [\gamma^2 u_{3,1} + iu_{3,2}]\mathrm{e}^{-i\theta}, \tag{5}$$

where $\gamma^2 = \mathrm{A}_{01313}/\mathrm{A}_{02323}$, $\hat{s}_{0r3} = s_{0r3}/\mathrm{A}_{02323}$ and $\hat{s}_{0\theta 3} = s_{0\theta 3}/\mathrm{A}_{02323}$.

2.2 Embedded Cavity and the Incident Plane SH-Wave

Consider an infinitely long circular cavity embedded in an unbounded pre-stressed elastic solid as shown in Fig. 1. The homogeneous principal stretches $\lambda_i, (i = 1, 2, 3)$ yield the corresponding homogeneous static principal Cauchy stresses $\sigma_i, (i = 1, 2, 3)$ (pg. 216, Ogden, 1984) and are given by,

$$\sigma_i = \lambda_i W_i / J, \quad (i = 1, 2, 3). \tag{6}$$

Since homogeneous principal stretches are assumed which yields homogeneous Cauchy stresses the internal static traction that should be applied along the inner surface of the cavity is

$$\mathbf{t}_0(\theta) = -(\sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta)\mathbf{e}_r + (\sigma_1 - \sigma_2)\cos\theta \sin\theta \, \mathbf{e}_\theta, \tag{7}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit basis vectors. For the equibiaxially pre-stressed $\sigma_1 = \sigma_2$ the traction $\mathbf{t}_0(\theta)$ in Eq. (7) corresponding to an internal static pressure (i.e., $p_0 = -\sigma_1$).

The incremental displacement of the incident time harmonic plane SH-wave $u_3^{(i)}(x_1, x_2, t) = U_0 \mathrm{e}^{-i\omega t} \mathrm{e}^{ik_\alpha (x_1 \cos\alpha + x_2 \sin\alpha)}$ can be expressed in the polar coordinate system as

$$u_3^{(i)}(r, \theta, t) = U_0 \mathrm{e}^{-i\omega t} \mathrm{e}^{ik_\alpha r \cos(\theta - \alpha)}, \tag{8}$$

where $\theta = \alpha$ is the direction of wave propagation, ω is an angular frequency, $k_\alpha = \omega / c_\alpha$ is wavenumber, and c_α is wave speed in this direction, i.e., $\rho c_\alpha^2 = \mathrm{A}_{01313} \cos^2\alpha + \mathrm{A}_{02323} \sin^2\alpha$ (pg. 474, Ogden 1984). Here the superscript (i) indicates the incident wave.

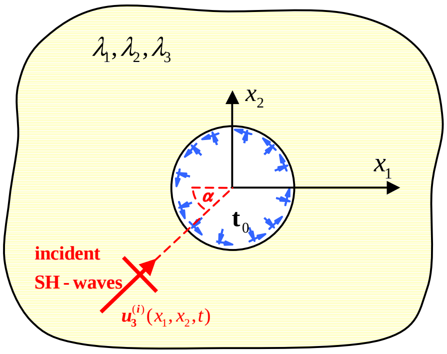

Figure 1. Unbounded pre-stressed material with cavity and the incident plane SH-wave.

Equation (8) may be expressed in the form of a Fourier series expansion (Liu, 1988) as,

$$u_3^{(i)}(r, \theta, t) = U_0 \mathrm{e}^{-i\omega t} \sum_{n=-\infty}^{\infty} i^n \mathrm{e}^{in(\theta - \alpha)} J_n(k_\alpha r) \tag{9}$$

where $J_n(k_\alpha r)$ is the Bessel function of order n with argument $k_\alpha r$.

To use the complex function method introduce the complex variables,

$$z = x_1 + i x_2 = r \mathrm{e}^{i\theta}, \quad \bar{z} = x_1 - i x_2 = r \mathrm{e}^{-i\theta}, \quad |z| = r, \tag{10}$$

which yields,

$$u_{3,1}^{(i)} = u_{3,z}^{(i)} + u_{3,\bar{z}}^{(i)}, \quad u_{3,2}^{(i)} = i(u_{3,z}^{(i)} - u_{3,\bar{z}}^{(i)}), \tag{11}$$

where $u_{3,z}^{(i)} = \partial u_3^{(i)} / \partial z$ and $u_{3,\bar{z}}^{(i)} = \partial u_3^{(i)} / \partial \bar{z}$, from which Eq. (9) can be expressed as

$$u_3^{(i)}(z, t) = U_0 \mathrm{e}^{-i\omega t} \sum_{n=-\infty}^{\infty} i^n \mathrm{e}^{-in\alpha} J_n(k_\alpha |z|)(z/|z|)^n. \tag{12}$$

From Eqs. (5), (11) and (12) the stress components due to the incident wave can be written as

$$\hat{s}^{(i)}_{0r3}(z,t) = \tfrac{1}{4}U_0 k_\alpha \,\mathrm{e}^{-i\omega t}\Big\{[(\gamma^2+1)\mathrm{e}^{i\theta}+(\gamma^2-1)\mathrm{e}^{-i\theta}]\sum_{n=-\infty}^{\infty} i^n\,\mathrm{e}^{-in\alpha}J_{n-1}(k_\alpha|z|)(z/|z|)^{n-1}$$
$$-[(\gamma^2-1)\mathrm{e}^{i\theta}+(\gamma^2+1)\mathrm{e}^{-i\theta}]\sum_{n=-\infty}^{\infty} i^n\,\mathrm{e}^{-in\alpha}J_{n+1}(k_\alpha|z|)(z/|z|)^{n+1}\Big\},$$
$$\hat{s}^{(i)}_{0\theta 3}(z,t) = \tfrac{i}{4}U_0 k_\alpha \,\mathrm{e}^{-i\omega t}\Big\{[(\gamma^2+1)\mathrm{e}^{i\theta}-(\gamma^2-1)\mathrm{e}^{-i\theta}]\sum_{n=-\infty}^{\infty} i^n\,\mathrm{e}^{-in\alpha}J_{n-1}(k_\alpha|z|)(z/|z|)^{n-1}$$
$$-[(\gamma^2-1)\mathrm{e}^{i\theta}-(\gamma^2+1)\mathrm{e}^{-i\theta}]\sum_{n=-\infty}^{\infty} i^n\,\mathrm{e}^{-in\alpha}J_{n+1}(k_\alpha|z|)(z/|z|)^{n+1}\Big\}. \tag{13}$$

2.3 Wave Scattering Solution

When the incident wave $u_3^{(i)}$ impinges on the surface of the cavity, the scattered wave $u_3^{(s)}$ is generated and the total displacement is the summation of both incident and scattered waves i.e., $u_3 = u_3^{(i)} + u_3^{(s)}$. For the scattered wave, another set of complex variables is introduced i.e.,

$$\eta = x_1 + i\gamma x_2 = r(\cos\theta + i\gamma\sin\theta), \qquad \bar{\eta} = x_1 - i\gamma x_2 = r(\cos\theta - i\gamma\sin\theta), \tag{14}$$

and hence,

$$u^{(s)}_{3,1} = u^{(s)}_{3,\eta} + u^{(s)}_{3,\bar{\eta}}, \quad u^{(s)}_{3,2} = i\gamma(u^{(s)}_{3,\eta} - u^{(s)}_{3,\bar{\eta}}),$$
$$u^{(s)}_{3,11} = u^{(s)}_{3,\eta\eta} + 2u^{(s)}_{3,\eta\bar{\eta}} + u^{(s)}_{3,\bar{\eta}\bar{\eta}}, \quad u^{(s)}_{3,22} = -\gamma^2(u^{(s)}_{3,\eta\eta} - 2u^{(s)}_{3,\eta\bar{\eta}} + u^{(s)}_{3,\bar{\eta}\bar{\eta}}). \tag{15}$$

Substituting Eq. (15) into Eq. (2a) yields

$$4c_0^2 u^{(s)}_{3,\eta\bar{\eta}} = \ddot{u}_3^{(s)}, \tag{16}$$

where c_0 is the SH-wave speed in x_1-direction i.e., $\rho c_0^2 = \mathrm{A}_{01313}$.

Following the work of Liu et al., (1982) the solution of Eq. (16), which satisfies the radiation condition when $r \to \infty$ can be written as

$$u_3^{(s)}(\eta,t) = \mathrm{e}^{-i\omega t}\sum_{n=-\infty}^{\infty} a_n H_n^{(1)}(k_0|\eta|)(\eta/|\eta|)^n, \tag{17}$$

where a_n, $n = 0, \pm 1, \pm 2, \ldots$ are arbitrary constants and $H_n^{(1)}(k_0|\eta|)$ is the Hankel function of order n with argument $k_0|\eta|$ and $k_0 = \omega/c_0$.

The corresponding stress components are

$$\hat{s}^{(s)}_{0r3}(\eta,t) = \tfrac{1}{4}k_0\,\mathrm{e}^{-i\omega t}\sum_{n=-\infty}^{\infty}\Big\{a_n[(\gamma^2+\gamma)\mathrm{e}^{i\theta}+(\gamma^2-\gamma)\mathrm{e}^{-i\theta}]H^{(1)}_{n-1}(k_0|\eta|)(\eta/|\eta|)^{n-1}$$
$$-[(\gamma^2-1)\mathrm{e}^{i\theta}+(\gamma^2+1)\mathrm{e}^{-i\theta}]H^{(1)}_{n+1}(k_0|\eta|)(\eta/|\eta|)^{n+1}\Big\},$$
$$\hat{s}^{(s)}_{0\theta 3}(\eta,t) = \tfrac{i}{4}k_0\,\mathrm{e}^{-i\omega t}\sum_{n=-\infty}^{\infty}\Big\{a_n[(\gamma^2+\gamma)\mathrm{e}^{i\theta}-(\gamma^2-\gamma)\mathrm{e}^{-i\theta}]H^{(1)}_{n-1}(k_0|\eta|)(\eta/|\eta|)^{n-1}$$
$$-[(\gamma^2-1)\mathrm{e}^{i\theta}-(\gamma^2+1)\mathrm{e}^{-i\theta}]H^{(1)}_{n+1}(k_0|\eta|)(\eta/|\eta|)^{n+1}\Big\}. \tag{18}$$

The incremental boundary condition along the surface of a circular cavity with radius a is expressed as

$$\hat{s}^{(i)}_{0r3}(z,t) + \hat{s}^{(s)}_{0r3}(\eta,t) = 0, \quad \text{on } |z| = a. \tag{19}$$

Substituting Eqs. (13) and (18) into Eq. (19) yields

$$\sum_{n=-\infty}^{\infty} a_n \phi_n = \phi, \tag{20}$$

where,

$$\begin{aligned}\phi_n &= (k_0 / k_\alpha)\Big\{[(\gamma^2+\gamma)\mathrm{e}^{i\theta}+(\gamma^2-\gamma)\mathrm{e}^{-i\theta}]H^{(1)}_{n-1}(k_0\,|\,\eta\,|)(\eta/|\,\eta\,|)^{n-1} \\ &\qquad -[(\gamma^2-1)\mathrm{e}^{i\theta}+(\gamma^2+1)\mathrm{e}^{-i\theta}]H^{(1)}_{n+1}(k_0\,|\,\eta\,|)(\eta/|\,\eta\,|)^{n+1}\Big\}, \\ \phi &= -[(\gamma^2+1)\mathrm{e}^{i\theta}+(\gamma^2-1)\mathrm{e}^{-i\theta}]\sum_{n=-\infty}^{\infty}(i)^n\,\mathrm{e}^{-in\alpha}\,J_{n-1}(k_\alpha\,|\,z\,|)(z/|\,z\,|)^{n-1} \\ &\qquad +[(\gamma^2-1)\mathrm{e}^{i\theta}+(\gamma^2+1)\mathrm{e}^{-i\theta}]\sum_{n=-\infty}^{\infty}(i)^n\,\mathrm{e}^{-in\alpha}\,J_{n+1}(k_\alpha\,|\,z\,|)(z/|\,z\,|)^{n+1}, \quad \text{on } |\,z\,|=a.\end{aligned} \tag{21}$$

Multiply both side of Eq. (20) with $\mathrm{e}^{-im\theta}$ and integrating from $-\pi$ to π yields a set of simultaneous equations:

$$\sum_{n=-\infty}^{\infty} A_n \phi_{mn} = \phi_m, \quad m = 0, \pm 1, \pm 2, \ldots \tag{22}$$

where $\phi_{mn} = \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi}\phi_n \mathrm{e}^{-im\theta} d\theta$ and $\phi_m = \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi}\phi\,\mathrm{e}^{-im\theta} d\theta$. The coefficients A_n, $n = 0, \pm 1, \pm 2, \ldots$ can be determined numerically by solving the above system of simultaneous equations.

2.4 Dynamic Stress Concentration Factor

Along the surface of the cavity, the dynamic stress concentration factor is defined as the ratio of incremental stress along the circumference to the maximum amplitude of the incident incremental stress at the same point. For the time harmonic incident SH-wave given in Eq. (8), the maximum incremental stress is $\left|s^{(i)}_{\max}\right| = \mathrm{A}_{02323}\gamma^2 k_\alpha U_0$. Therefore the dynamic stress concentration factor is

$$\left|s_{0\theta 3}/s^{(i)}_{\max}\right| = |\,\hat{s}^{(i)}_{0\theta 3} + \hat{s}^{(s)}_{0\theta 3}\,| / (\gamma^2 k_\alpha U_0). \tag{23}$$

3. NUMERICAL RESULTS

As mention in Sec. 2.1 the instantaneous elastic moduli A_{01313} and A_{02323} depend on the strain energy function of the material and the principal stretches. In this section compressible Varga material is assumed and the strain energy function is given as (Roxburgh and Ogden, 1994),

$$W^{(V)} = 2\mu_0[\lambda_1 + \lambda_2 + \lambda_3 - 3 - \ln(\lambda_1\lambda_2\lambda_3)]. \tag{24}$$

From the definition of A_{01313}, A_{02323}, and γ in Sec. 2.1, Eq. (24) yields

$$J\mathrm{A}_{01313} = 2\mu_0\lambda_1^2/(\lambda_1+\lambda_3), \qquad J\mathrm{A}_{02323} = 2\mu_0\,\lambda_2^2/(\lambda_2+\lambda_3), \qquad \gamma = \frac{\lambda_1}{\lambda_2}\sqrt{\frac{\lambda_2+\lambda_3}{\lambda_1+\lambda_3}}. \tag{25}$$

In the absence of pre-stress (i.e., $\lambda_1 = \lambda_2 = \lambda_3 = 1$), Eq. (25) yields $\mathrm{A}_{01313} = \mathrm{A}_{02323} = \mu_0$, $\gamma = 1$ which from the orthogonal properties of $\mathrm{e}^{-im\theta}$, $m = 0, \pm 1, \pm 2, \ldots$ the linear isotropic solution (pg. 121, Pao and Mow, 1973) is recovered and the coefficient A_n in Eq. (22) can be expressed as

$$A_n = -i^n \frac{nJ_n(k_0a_0) - k_0a_0 J_{n+1}(k_0a_0)}{nH_n^{(1)}(k_0a_0) - k_0a_0 H_n^{(1)}(k_0a_0)}, \qquad (n = 0, \pm 1, \pm 2, \ldots), \tag{26}$$

where in this case $k_0 = \omega\sqrt{\rho_0/\mu_0}$ is the wavenumber, ρ_0 and μ_0 are the material density and the shear stiffness in an undeformed configuration, respectively.

For the pre-stressed material that is equibiaxially deformed in (x_1x_2)-plane (i.e., $\lambda_1 = \lambda_2 = \lambda$), Eq. (25) yields $\mathrm{A}_{01313} = \mathrm{A}_{02323} = \mu$ and $\gamma = 1$, where $\mu = 2\mu_0\lambda_1^2 / J(\lambda_1 + \lambda_3)$ is the shear stiffness of the material in the equilibrium configuration. Since $\gamma = 1$ the coefficient A_n is still given by Eq. (26) but with k_0a_0 replaced by $ka = \omega a\sqrt{\rho/\mu}$ where $\rho = \rho_0 / J$ and $a = a_0\lambda_1^2$ are material density and radius of cavity in the equilibrium configuration, respectively. It can be seen that for the same frequency of the incident waves, the dynamic stress concentration factor $\left|s_{0\theta 3} / s_{\max}^{(i)}\right|$ in pre-stress media is different with of linear isotropic case.

Example 1: Figure 2 shows the geometry of this example when the material is equibiaxially deformed in (x_1x_2)-plane (i.e., $\lambda_1 = \lambda_2 = \lambda$), the internal pressure $p_0 = 2\mu_0(1-\lambda)/J$ is applied inside the cavity. The internal pressure is necessary since homogeneous stretches are assumed. The incident wave is assumed to propagate in the x_1-direction with non-dimensional frequency $\varpi = \omega\sqrt{\rho_0 a_0^2 / \mu_0}$. The principal stretches are varying as $\lambda = 0.1, 0.5, 1.0$ and $\lambda_3 = 0.5, 1.0, 2.0$. The non-dimensional phase speed of the SH-wave $\bar{c} = \sqrt{\mu\rho_0 / \mu_0\rho}$ and non-dimensional internal pressure $\bar{p} = p_0/\mu_0$ which depend on the values of principal stretches are shown in Fig. 3, while the dynamic stress concentration factors are shown in Fig. 4.

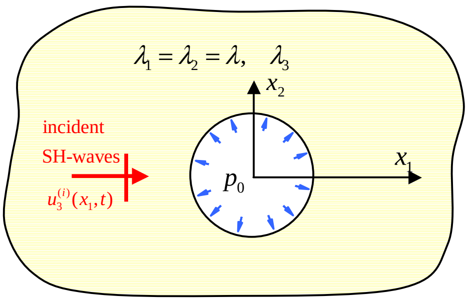

Figure 2. Geometry of Example 1.

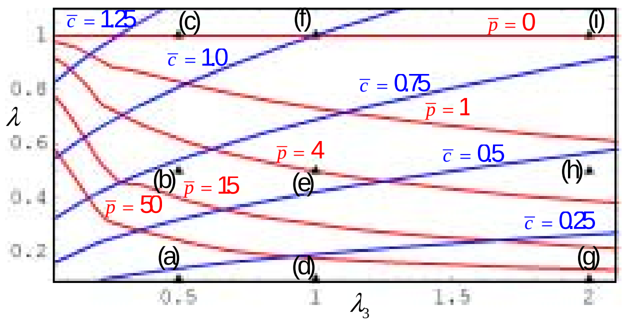

Figure 3. Contour plot of non-dimensional frequency $\bar{c}$ and pressure $\bar{p}$.

It can be seen in Fig. 3 that for $\lambda_3 < \lambda < 1.0$ which simulates the pre-stressed earth when the (x_1x_2)-plane is parallel to the horizontal ground surface, the speed of SH-waves is slower than that of the non pre-stressed material i.e., $\bar{c} < 1.0$, while the surrounding pressure $\bar{p} > 0$.

Since the (x_1x_3)-plane is a plane of symmetry $\left|s_{0\theta 3} / s_{\max}^{(i)}\right|$ is plotted for $0^\circ \le \theta \le 180^\circ$. It should be note that the maximum static stress concentration factor in the linear isotropic case is 2.0 at $\theta = 90^\circ$ (pg. 133, Pao and Mow, 1973). In this example when $\varpi = 0.1$ the maximum value of $\left|s_{0\theta 3} / s_{\max}^{(i)}\right|$ is slightly greater than 2.0 and for the higher frequencies ($\varpi = 1, 2$) the maximum values of $\left|s_{0\theta 3} / s_{\max}^{(i)}\right|$ are lower than 2.0. Therefore, the low frequency wave is an important consideration in the study of dynamic stress concentrations in pre-stress media as well as in linear elastic material. The effect of pre-stress on the distribution of $\left|s_{0\theta 3} / s_{\max}^{(i)}\right|$ is more for high frequency and less for the low frequency waves.

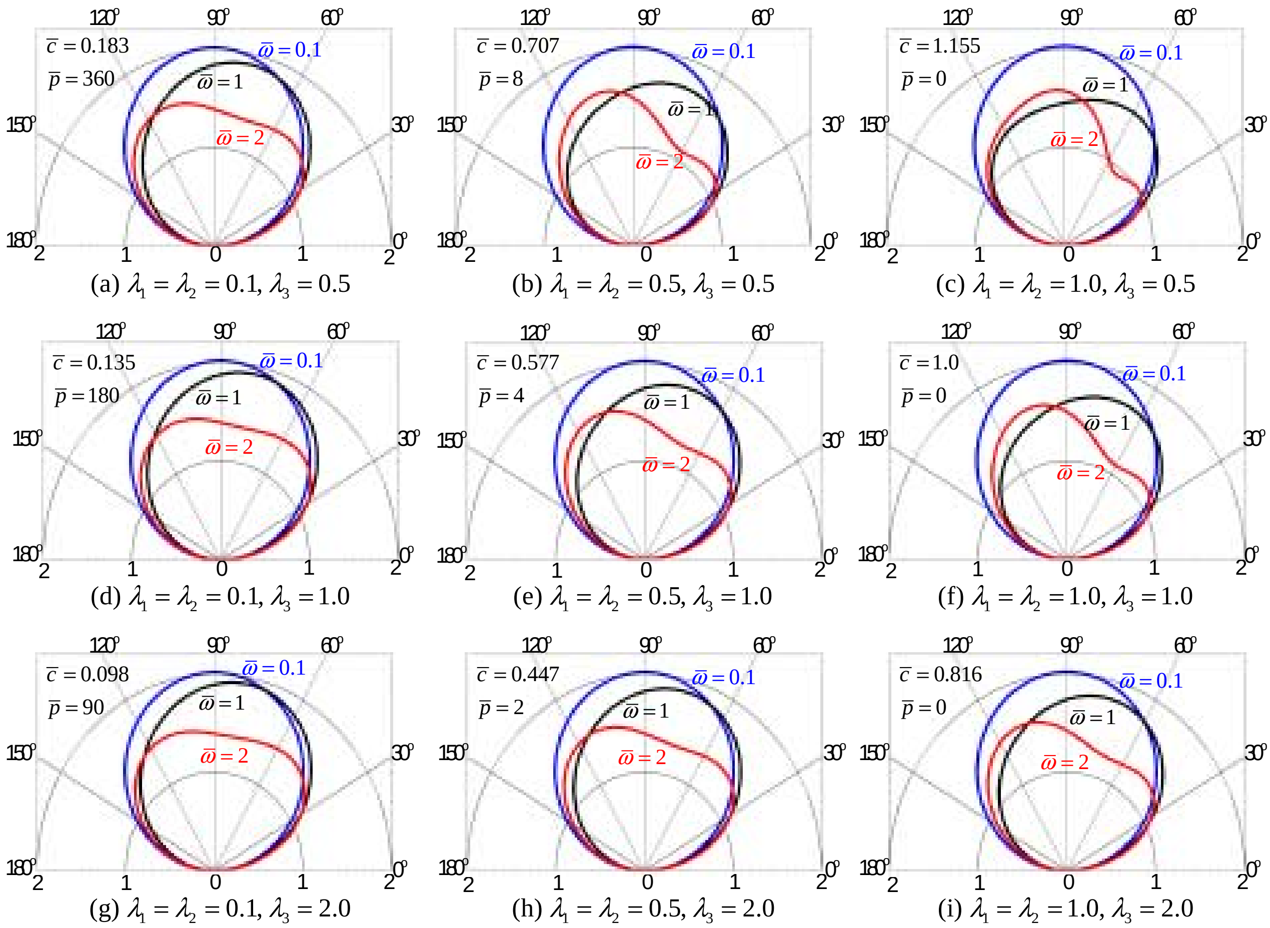

Figure 4. Dynamic stress concentration factor $\left|s_{0\theta 3}/s_{\max}^{(i)}\right|$ of Example 1.

Example 2: The Varga material is equibiaxially deformed in the (x_1x_3)-plane with the principal stretches $\lambda_1=\lambda_3=\lambda=0.9$ and $\lambda_2=0.7$, where the internal static traction $\mathbf{t}_0(\theta)=3.527\mu_0$ $[(3\cos^2\theta+\ \sin^2\theta)\mathbf{e}_r-\sin 2\theta\,\mathbf{e}_\theta]$ is applied along the inner surface of the cavity since uniform stretches are assumed. The incident SH-wave has an incident angle $\alpha=45^{\circ}$ (see Fig. 1) and non-dimensional frequency $\omega\sqrt{\rho a^2/\mu}=0.5$. Using the method presented in Sec. 2 with $n_{\max}=20$, the coefficients A_n, $n=0,\pm1,\pm2,...,\pm20$ are numerically obtained. The real and imaginary parts of the amplitude of the displacement and shear stresses along the surface of cavity and the distribution of dynamic stress concentration are plotted in Figs. 5-8.

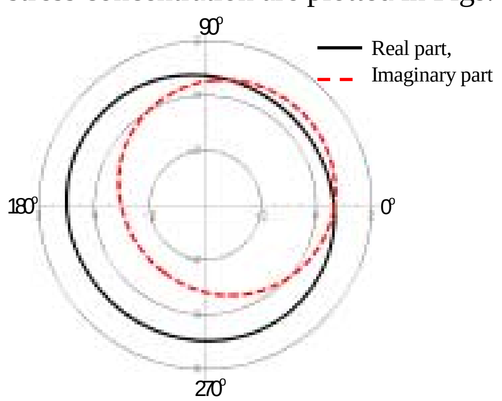

Figure 5. Non-dimensional displacement u_3/U_0 along the surface of cavity.

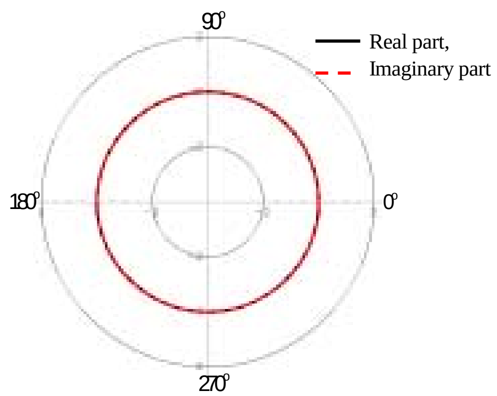

Figure 6. Non-dimensional shear stress $s_{0r3}/|s_{\max}^{(i)}|$ along the surface of cavity.

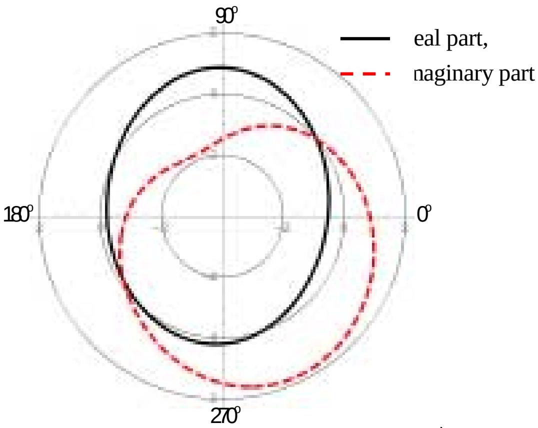

Figure 7. Dynamic stress concentration $s_{0\theta 3}/|s_{\max}^{(i)}|$.

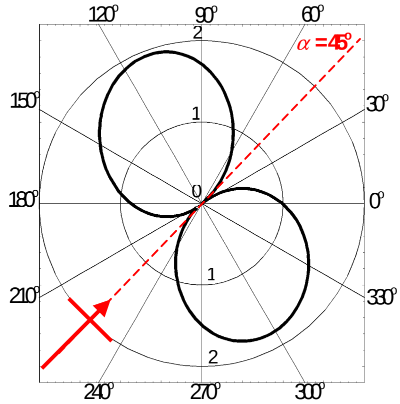

Figure 8. Dynamic stress concentration factor $\left|s_{0\theta 3}/s_{\max}^{(i)}\right|$.

Figures 5 and 7 can be used to calculate the displacement and shear stresses at any time in the period of vibration, while Fig. 6 shows that the results agree with the boundary condition at the surface of the cavity. It is seen from Fig. 8 that the position of maximum stress concentration factor is at $\theta = 110.1^{\circ}$ with $\left|s_{0\theta 3}/s_{\max}^{(i)}\right| = 1.939$, and the another local maximum is at $\theta = 297.3^{\circ}$ with $\left|s_{0\theta 3}/s_{\max}^{(i)}\right| = 1.810$ and the distribution of $\left|s_{0\theta 3}/s_{\max}^{(i)}\right|$ is not symmetric with respect to any plane or axis for this example.

4. CONCLUSIONS

Using the complex function method the scattering of plane SH-waves from a circular cylindrical cavity in a pre-stressed elastic medium is analyzed. The effect of pre-stress on the speed of plane SH-waves and the dynamic stress concentration factor can be clearly seen from the numerical results. Low frequency waves will have a higher stress concentration than the high frequency waves. The distribution of stress concentration factor for pre-stressed media is not always symmetric, except for the in-plane equibiaxially deformed case. Scattering problems for non-circular cavities and inclusions in pre-stressed elastic unbounded media or in a half-space and the scattering of in-plane waves (P and SV waves) should also be studied.

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